MATHEMATICS

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AIM POINTS
MATHEMATICS
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XIth, XIIth, TARGET IIT-JEE (MAIN + ADVANCE) & COMPETITIVE EXAM. FOR XI (PQRS)

COMPLEX NUMBERS
& Their Properties

CONTENTS

THINGS TO REMEMBER

1. (i) $\left(\bar{z}\right) = z$

   (ii) $z + \bar{z} = 2\ \text{Re}\ (z)$

   (iii) $z - \bar{z} = 2\ i\ \text{Im}\ (z)$

   (iv) $z = \bar{z} \Leftrightarrow z$ is purely, real

   (v) $z + \bar{z} = 0 \Rightarrow z$ is purely imaginary

   (vi) $z\bar{z} = \{\text{Re}\ (z)\}^2 + \{\text{Im}\ (z)\}^2$

   (vii) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

   (viii) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

   (ix) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

   (x) $\left(\frac{z}{z}\right) = \frac{\overline{z_1}}{z_2}, z_2 \neq 0$

2. (i) $|z| = 0 \Leftrightarrow z = 0$ i.e. Re (z) = Im (z) = 0

   (ii) $|z| = |\bar{z}| = |-z|$

   (iii) $-|z| \leq \text{Re}\ (z) \leq |Z|; -|z| \leq \text{Im}\ (z) \leq |z|$

   (iv) $z\bar{z} = |z|^2$

   (v) $|z_1 z_2| = |z_1|\,|z_2|$

   (vi) $\left|\frac{z_1}{z_2}\right| = \left|\frac{z_1}{z_2}\right|; z_2 \neq 0$

   (vii) $|z_1 + z_2|^2 = |z_1| + |z_2|^2 + 2\text{Re}\left(z_1\overline{z_2}\right)$

   (viii) $|z_1 - z_2|^2 = |z_1| + |z_2|^2 - 2\text{Re}\left(z_1\overline{z_2}\right)$

   (ix) $|z_1 + z_2|^2 = |z_1 - z_2|^2 = (|z_1|^2 + |z_2|^2)$

   (x) $|az_1 - bz_2|^2 = |bz_1 + bz_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$, where $a, b \in R$.

3. $\sqrt{-1}$ is an imaginary quantity and is denoted by i which has the following properties :

   $i^2 = -1, i^3 = -i, i^4 = 1$ and, $i^{\pm n} + i^{\pm k}, n \in N$

   where k is the remainder when n is denoted by 4.

4. For any positive real number a, $\sqrt{-a} = i\sqrt{a}$

5. For any two real numbers a and b, we have

$$\sqrt{a}\sqrt{b} = \begin{cases} \sqrt{ab} & \text{, if at least one of a and b is positive} \\ -\sqrt{ab} & \text{, if } a < 0, b < 0 \end{cases}$$

6. If a, b are real numbers, then a numer z = a + ib is called a complex number. Real number a is known as the real part of z and b is known as its imaginary part.

   We write a = Re (z), b = Im (z)

   A complex number z is purely real iff Im (z) = 0 and z is purely imaginary iff Re (z) = 0

7. For any two complex numbers $z_1 + ib_1$ and $z_2 = a_2 + ib_2$, we define

   Addition : $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$

   Subtraction : $z_1 - z_2 = (a_1 - a_2) - i(b_1 - b_2)$

   Multiplication : $z_1z_2 = (a_1a_2 - b_1b_2) + i(a_1a_2 + b_1b_2)$

   Reciprocal : $\dfrac{1}{z_1} = \dfrac{a_1}{a_1^2 + b_1^2} - i\dfrac{b_1}{a_1^2 + b_1^2}$

   Division : $\dfrac{z_1}{z_2} = z_1\left(\dfrac{1}{z_2}\right) = (a_1 + ib_1)\left(\dfrac{a_2}{a_2^2 + b_2^2} - i\dfrac{b_2}{a_2^2 + b_2^2}\right)$

   $$= \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\frac{a_1b_1 - a_1b_1}{a_2^2 + b_2^2}$$

   Addition is commutative and associative. Complex number 0 = 0 + i0 is the identity element for addition and every complex number z = a + ib has its additive inverse – z = – a – ib.

   Multiplication is also commutative and asosciative. Complex number 1 = 1 + 0i is the identity element for multiplication. Every non–zero complex number z = a + ib has its multiplicative inverse 1/z (also known as reciprocal of z) such that

   $$\frac{1}{z} = \frac{a - ib}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$$

8. The conjugate of a complex number z = a + ib is denoted by $\bar{z}$ and is equal to ~~i – ib.~~ a – ib

EXERCISE-1

1. Show that :

   (i) $\left\{i^{19} + \left(\dfrac{1}{i}\right)^{25}\right\}^2 = -4$

   (i) $\left\{i^{17} + \left(\dfrac{1}{i}\right)^{34}\right\}^2 = 2i$

   (iii) $\left\{i^{18} + \left(\dfrac{1}{i}\right)^{24}\right\}^3 = 0$

   (iv) $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$ for all $n \in N$

2. Show that $1 + i^{10} + i^{20} + i^{30}$ is a real number.

3. Find the values of the following expressions :

(i) $i^{49} + i^{68} + i^{89} + i^{110}$ (ii) $i^{30} + i^{80} + i^{120}$

(iii) $i + i^2 + i^3 + i^4$ (iv) $i^5 + i^{10} + i^{15}$

(v) $\dfrac{i^{592} + i^{590} + i^{588} + i^{566} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}}$ (vi) $1 + i^2 + i^4 + i^6 + i^8 + ... + i^{20}$

4. Properties of Multiplication.

5. Reciprocal of a complex number.

6. Express each of the following in the form a + ib :

(i) $3(7 + 7i) + i(7 + 7i)$ (ii) $(1 - i) = (-1 + 6i)$

(iii) $\left(\frac{1}{5}+\frac{2}{5}i\right)-\left(4+\frac{5}{2}i\right)$ (iv) $\left\{\left(\frac{1}{3}+\frac{7}{3}i\right)+\left(4+\frac{1}{3}i\right)-\left(-\frac{4}{3}+i\right)\right\}$

7. Express each of the following in the form a + ib :

(i) $\left(\frac{1}{3}+3i\right)^3$ (ii) $\left(-2-\frac{1}{3}i\right)^3$

(iii) $(5 - 3i)^3$ (iv) $\left(-\sqrt{3}+\sqrt{-2}\right)\left(2\sqrt{3}-i\right)$

8. If $a + ib = \dfrac{c+i}{c-i}$, where c is real, prove that : $a^2 + b^2 = 1$ and $\dfrac{b}{a}=\dfrac{2c}{c^2-1}$

9. Find real values of x and y for which the complex numbers $-3 + ix^2 y$ and $x^2 + y + 4i$ are conjugate of each other.

10. Find the multiplicative inverse of the following complex numbers :

(i) $3 + 2i$ (ii) $\left(2+\sqrt{3}i\right)^2$

11. If $\dfrac{(a+i)^2}{(2a-i)}=p+iq$, show that : $p^2 + q^2 = \dfrac{(a^2+1)^2}{(4a^2+1)}$

12. If $x = -5 + 2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.

13. If z is a complex number such that | z | = 1, prove that $\left(\dfrac{z-1}{z+1}\right)$ is purely imaginary. What will be your conclusion if z = 1 ?

14. If z = 2 – 3i, show that $z^2 - 4z + 13 = 0$ and hence find the value of $4z^3 - 3z^2 + 169$.

15. If α and β are different complex numbers with | β | = 1, find $\left|\dfrac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|$.

16. Find non–zero integral solutions of $|1 - i|^x = 2^x$.

17. Find all non–zero complex numbers z satisfying $\bar{z} = i\,z^2$.

18. Find the real value of x and y, if

(i) $(x + i\,y)(2 - 3i) = 4 + i$

(ii) $(3x - 2i\,y)(2 + i)^2 = 10(1 + i)$

(iii) $\dfrac{(1+i)x - 2i}{3+i} + \dfrac{(2-3i)y + i}{3-i} = i$

(iv) $(1 + i)(x + i\,y) = 2 - 5i$

19. Evaluate the following :

(i) $3x^3 + 2x^2 - 7x + 72$, when $x = \dfrac{3-5i}{2}$

(ii) $x^4 - 4x^3 + 4x^2 + 8x + 44$, when $x = 3 + 2i$

(iii) $x^4 + 4x^3 + 6x^2 + 4x + 9$, when $x = -1 + i\sqrt{2}$

(iv) $x^6 + x^4 + x^2 + 1$, when $x = \dfrac{1+i}{\sqrt{2}}$

20. If $z_1 = 2 - i$, $z_2 = -2 + i$, find

(i) $\text{Re}\left(\dfrac{z_1 z_2}{\bar{z}_1}\right)$

(ii) $\left(\dfrac{1}{z_1 \bar{z}_1}\right)$

21. Find the least positive integral value of n for which $\left(\dfrac{1+i}{1-i}\right)^n$ is real.

22. Find the smallest positive integer value of n for which $\dfrac{(1+i)^n}{(1-i)^{n-2}}$ is a real number.

23. Find the square roots of the following :

(i) $7 - 24i$

(ii) $5 + 12i$

25. Find the square roots of $-15 - 8i$.

26. Find the square root of i.

27. Find the modulus and argument of each of the following complex numbers :

(i) $1 + i\sqrt{3}$

(ii) $-2 + 2\,i\sqrt{3}$

(iii) $-\sqrt{3} - i$

(iv) $2\sqrt{3} - 2\,i$

28. Find the modulus and argument of the following complex numbers :

(i) $\dfrac{1+i}{1-i}$

(ii) $\dfrac{1}{1+i}$

29. Find the modulus and principal argument of –4.

30. Put the following in the form $r(\cos\theta + i\sin\theta)$, where r is a positive real number and $-\pi < \theta \le \pi$: $\dfrac{(1+7i)}{(2-i)^2}$

31. Multiplication of a complex number by ~~iota.~~ Mode .

EXERCISE-2

Mark the correct alternative in each of the following

1. If $z = \cos\frac{\pi}{4} + r\sin\frac{\pi}{6}$, then

(a) $|z| = 1, \arg(z) = \frac{\pi}{4}$ (b) $|z| = 1, \arg(z) = \frac{\pi}{6}$

(c) $|z| = \frac{\sqrt{3}}{2}, \arg(z) = \frac{5\pi}{24}$ (d) $|z| = \frac{\sqrt{3}}{2}, \arg(z) = \tan^{-1}\frac{1}{\sqrt{2}}$

2. If $i^2 = -1$, then the sum $i + i^2 + i^3 + \ldots$ upto 1000 terms in equal to

(a) 1 (b) –1 (c) i (d) 0

3. If $\frac{(a^2+1)^2}{2a-i} = x + iy$, then $x^2 + y^2$ is equal to

(a) $\frac{(a^2+1)^4}{4a^2+1}$ (b) $\frac{(a+1)^2}{4a^2+1^2}$ (c) $\frac{(a^2-1)^2}{(4a^2-1)^2}$ (d) none of these

4. If z is a non–zero complex number, then $\left|\frac{|\bar{z}|^2}{z\bar{z}}\right|$ is equal to

(a) $\left|\frac{\bar{z}}{z}\right|$ (b) $|z|$ (c) $|\bar{z}|$ (d) none of these

5. $\left(\sqrt{-2}\right)\left(\sqrt{-3}\right)$ is equal to

(a) $\sqrt{6}$ (b) $-\sqrt{6}$ (c) $i\sqrt{6}$ (d) none of these

6. If $z = \left(\frac{1+i}{1-i}\right)$, then z^4 equals

(a) 1 (b) –1 (c) 0 (d) none of these

7. If $z = \frac{1}{(10i)(2+3i)}$, then $|z| =$

(a) 1 (b) $\frac{1}{\sqrt{26}}$ (c) $\frac{5}{\sqrt{26}}$ (d) none of these

8. If $\frac{1-ix}{1+ix} = a + ib$, then $a^2 + b^2 =$

(a) 1 (b) –1 (c) 0 (d) none of these

9. If $z = \frac{1+7i}{(2-i)^2}$, then

(a) $|z| = 2$ (b) $|z| = \frac{1}{2}$ (c) amp $(z) = \frac{\pi}{4}$ (d) amp $(z) = \frac{3\pi}{4}$

10. The argument of $\frac{1-i}{1+i}$ is

(a) $-\frac{\pi}{2}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) $\frac{5\pi}{2}$

11. The value of $\frac{(i^5+i^6+i^7+i^8+i^9)}{(1+i)}$ is

(a) $\frac{1}{2}(1+i)$ (b) $\frac{1}{2}(1-i)$ (c) 1 (d) $\frac{1}{2}$

12. The value of $(1+i)^4 + (1-i)^4$ is

(a) 8 (b) 4 (c) –8 (d) –4